

Einstein and Brans-Dicke frames in multidimensional cosmology

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Abstract

Inhomogeneous multidimensional cosmological models with a higher dimensional space-time manifold $M = \overline{M}_0 \times \prod_{i=1}^n M_i$ ($n \geq 1$) are investigated under dimensional reduction to a D_0 -dimensional effective non-minimally coupled σ -model which generalizes the familiar Brans-Dicke model. It is argued that the Einstein frame should be considered as the physical one. The general prescription for the Einstein frame reformulation of known solutions in the Brans-Dicke frame is given. As an example, the reformulation is demonstrated explicitly for the generalized Kasner solutions where it is shown that in the Einstein frame there are no solutions with inflation of the external space.

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I. INTRODUCTION

All contemporary unified interaction models face the requirement also to incorporate gravity. The most prominent attempt in this direction is string theory and its recent extension of M-theory [1,2] which extends strings to generalized membranes as higher-dimensional objects. Most of these unified models are modeled initially on a higher-dimensional space-time manifold, say of dimension $D > 4$, which then undergoes some scheme of spontaneous compactification yielding a direct product manifold $M^4 \times K^{D-4}$ where M^4 is the manifold of space-time and K^{D-4} is a compact internal space (see e.g. [3]- [6]). Hence it is natural to investigate cosmological consequences of such a hypothesis.

In particular we will investigate multidimensional cosmological models (MCM) given as a topological product

$$M = \overline{M}_0 \times \prod_{i=1}^n M_i, \quad (1.1)$$

where $\overline{M}_0 := \mathbb{R} \times M_0$ is a D_0 -dimensional (usually $D_0 = 4$) smooth space-time manifold with spatial sections all diffeomorphic to a standard section M_0 , and $\prod_{i=1}^n M_i$ an internal product space from smooth homogeneous factor spaces M_i of dimension d_i , $i = 1, \dots, n$.

Let \overline{M}_0 be equipped with a smooth hyperbolic metric $\overline{g}^{(0)}$, let $\overline{\gamma}$ and β^i , $i = 1, \dots, n$ be smooth scalar fields on \overline{M}_0 , and let each M_i be equipped with a smooth homogeneous metric $g^{(i)}$. Then, under any projection $\text{pr} : M \rightarrow \overline{M}_0$ a pullback consistent with (1.1) of $e^{2\overline{\gamma}}\overline{g}^{(0)}$ from $x \in \overline{M}_0$ to $z \in \text{pr}^{-1}\{x\} \subset M$ is given by

$$g(z) := e^{2\overline{\gamma}(x)}\overline{g}^{(0)}(x) + \sum_{i=1}^n e^{2\beta^i(x)}g^{(i)}. \quad (1.2)$$

The function $\overline{\gamma}$ fixes a *gauge* for the (Weyl) *conformal frame* on \overline{M}_0 . Note that the latter has little in common with a usual (coordinate) frame of reference. Rather it corresponds to a particular choice of geometrical variables, whence it might also be called a (classical) *representation* of the metric geometry. All these terms are often used synonymously in the literature, and so we do below.

We will show below how $\overline{\gamma}$ uniquely defines the form of the effective D_0 -dimensional theory. For example $\overline{\gamma} := 0$ defines the Brans-Dicke frame¹ with a non-minimally coupled dilatonic² scalar field given by the total internal space volume, while (for $D_0 \neq 2$) $\overline{\gamma} := \frac{1}{2-D_0} \sum_{i=1}^n d_i \beta^i$ defines the Einstein frame³ with all dilatonic scalar fields minimally coupled.

There is a long and still ongoing (see e.g. [7]) discussion in the literature which frame is the physical one. Historical references on this subject are contained in [8], and more recent ones in [9].

From the mathematical point of view, the equivalence of all classical representations of smooth geometrical models based on multidimensional metrics (1.2) related by different choices of the smooth gauge function $\overline{\gamma}$ is guaranteed by the manifest regularity of the conformal factor $e^{2\overline{\gamma}} > 0$. Hence the spaces of regular and smooth local classical solutions are isomorphic for all regular and smooth representations of the classical geometrical theory. Note however that physically interesting choices of $\overline{\gamma}$ might sometimes fail to exist within any class of functions which satisfies the required regularity and smoothness conditions. So e.g. for $D_0 = 2$ a gauge of $\overline{\gamma}$ yielding the Einstein frame fails to exist, whence some 2-dimensional scalar-tensor theory obtained by dimensional reduction from a multidimensional geometry is in general not conformally equivalent to a theory with minimal coupling.

Even if two classical conformal representations are equivalent from the purely geometrical point, their different coupling of a dilatonic scalar field to the metric geometry in different conformal frames distinguishes the representations physically, if and only if physics depends indeed on the metric geometry rather than on the Weyl geometry only.

Moreover, if the theory incorporates additional matter fields, the dynamics of these fields

¹This frame is sometimes also called Brans-Dicke-Jordan frame, or simply Jordan frame.

²Here by a *dilatonic* scalar field we refer to any scalar field which is given in terms of logarithms of internal space scale factors.

³This frame is sometimes also called Einstein-Pauli frame, or simply Pauli frame.

may reveal the true physical frame to which they couple.

Let us point out in advance the main advantages of the Einstein frame, corresponding to $f = 0$ in (2.5) below, for the multidimensional model (1.1) with metric (1.2).

First, in this frame all dilatonic fields have the same (positive) sign in all kinetic terms. In other frames with $f \neq 0$ there is a dilatonic kinetic term, $(\partial f)^2$ in (2.15), which may have an opposite (negative) sign corresponding to a ghost. Hence there is no way to guarantee “unitarity” (i.e. the positive definiteness of the Hamiltonian) for the action (2.15) if one tries to identify a Brans-Dicke frame (with $f \neq 0$) as a physical one [10]. Although the gauge $f = -2\bar{\gamma}$ also provides the correct sign for all kinetic terms, it has another drawback in its coupling to additional scalar matter fields (see below) .

Secondly, Cho [10] has also shown that only in the Einstein frame the perturbative part of the gravitational interaction is generated purely by spin-2 gravitons. In any Brans-Dicke frame additional spin-0 scalar particles enter as basic perturbative modes of gravity.

Third, only in the Einstein frame the D_0 -dimensional effective gravitational constant (which is Newton’s constant for $D_0 = 4$) is an exact constant, such that the present day experimental bounds on the variation of the gravitational constant [11,12] are solved automatically, while in all Brans-Dicke frames fine-tuning is necessary.

Related arguments in favor of the Einstein frame given in [8] for 4-dimensional non-linear (higher order) gravitational models may be applied analogously to scalar-tensor gravity theories as the ones considered here. There it was shown that “the existence of the Einstein frame is in any case essential for assessing classical stability of Minkowski space and positivity of energy for nearby solutions. In the Jordan frame, the dominant energy condition never holds. For these reasons, the Einstein frame is the most natural candidate for the role of physical frame”. Note that while [8] discussed a generic possibility for (multi-)scalar-tensor theories to couple extra scalar fields at hand to the dilatonic one and to the metric in any

conformal frame for our D_0 -dimensional effective theory this possibility does not arise.⁴ Here a choice of gauge function $\overline{\gamma}$ not only fixes a conformal frame and the dilatonic field f but also all the couplings with further dilatonic scalar fields, prescribed then by the particular multidimensional structure of our D -dimensional theory.

Below we take this higher-dimensional theory in form of an Einsteinian theory plus any minimally coupled D -dimensional matter which is in accordance with ansatz given by (1.1) and (1.2) homogeneous in the $(D - D_0)$ - dimensional internal space, i.e. all free functions only depend on \overline{M}_0 (like e.g. the zero mode fields in [14]).

With the above multidimensional structure, this ansatz fixes also all couplings of this extra matter to geometrical fields in the effective D_0 -dimensional theory. It is evident from (2.15) below that the extra matter is also minimally coupled to $\overline{g}^{(0)}$ for any gauge of $\overline{\gamma}$, but its coupling to the dilatonic field strongly varies with $\overline{\gamma}$.

In a Brans-Dicke frame (where $\overline{\gamma} = 0$) matter couples directly to a dilatonic prefactor $e^f = \prod_{i=1}^n a_i^{d_i}$ (in front of kinetic as well as potential terms) which is proportional to the Riemann-Lebesgue volume of the total internal space.

In the Einstein frame ($f = 0$) dilatonic fields become, like the extra matter, minimally coupled to the geometry of \overline{M}_0 . Then, the extra matter is coupled to dilatonic fields by via a potential term of the effective D_0 -dimensional theory.

So, in the Einstein frame the physical setting is rather clear:

First, with respect to scalar fields of dilatonic origin the theory has the shape of a self-gravitating σ -model [9] with flat Euclidean target space and self-interaction described by an effective potential. Eventually existing minima of this potential have been identified as

⁴ In this aspect our approach differs also essentially from that of [13], who do not consider scalar fields and their couplings as given by reduction from a higher-dimensional space, whence from that point of view it is still consistent when they favor a Brans-Dicke frame.

positions admitting a stable compactification.⁵ Small fluctuations of scale factors of the internal spaces near such minima could in principle be observed as massive scalar fields (gravitational excitons) in the external space-time [17].

Second, under the assumption that the fluctuations of the internal space scale factors around a stable position at one of the minima mentioned above are very small, the extra matter fields (of any type) might be considered in an approximation of order zero in these fluctuations. In this approximation they have the usual free D_0 -dimensional form and follow the geodesics of the metric $g^{(E)} := \bar{g}^{(0)}|_{f=0}$. Taking into account the first nontrivial order in these fluctuations yields the gravitational excitons plus an interaction between the extra matter and the excitons [14]. A likewise clear structure is not at hand for the corresponding theory in a Brans-Dicke frame.

Besides these arguments in favor of the Einstein frame, it is interesting to note that many investigations of astrophysical consequences for scalar-tensor theories are also performed in the Einstein frame as the physical one [18–20].

For cosmological models with multidimensional structure (1.1) and metric structure (1.2) most exact solutions of the field equations were obtained in the spatially homogeneous case, where the scale factors $a_i := e^{\beta^i}$, $i = 1, \dots, n$ are only a function of time $t \in \mathbb{R}$. Some overview and an extensive list of references is given in [21–24]. All solutions known to us have been obtained exploiting the simple coupling ($\bar{\gamma} = 0$) in the Brans-Dicke frame. However the arguments above show that these solutions should be reformulated in the Einstein frame before a physical interpretation is given. It is clearly to be expected that the reinterpreted solutions will have a different qualitative behavior as compared to those in the Brans-Dicke frame. The concretization of this expectation is our major motivation for

⁵The stability analysis of the compactified internal spaces in multidimensional cosmological models [15] as well as multidimensional black hole solutions [16] has also been performed in the Einstein frame.

the present investigations. The common underlying structure of many exact solutions rises the possibility to find an explicit description of the transition from Brans-Dicke to Einstein frame for rather general classes of solutions.

As an important example, the exact transformation can be performed for the well known generalized Kasner solution. The so obtained solution in the Einstein frame is indeed qualitatively quite different than the Kasner one, which conclusively supports our previous expectations.

The paper is organized as follows: In Sec. II we describe the multidimensional model and obtain a dimensionally reduced effective theory in an arbitrary frame. Sec. III presents a general method for transformations between solutions in Brans-Dicke and Einstein frames. A brief review of the generalized Kasner solution in the Brans-Dicke frame is given in Sec. IV. Its explicit reformulation in the Einstein frame follows in Sec. V.

II. MULTIDIMENSIONAL GEOMETRY AS EFFECTIVE σ -MODEL

Let us now consider a multidimensional manifold (1.1) of dimension $D = D_0 + \sum_{i=1}^n d_i = 1 + \sum_{i=0}^n d_i$, equipped with a (pseudo) Riemannian metric (1.2) where

$$g^{(i)} \equiv g_{m_i n_i}(y_i) dy_i^{m_i} \otimes dy_i^{n_i}, \quad (2.1)$$

are R -homogeneous Riemannian metrics on M_i (i.e. the Ricci scalar $R[g^{(i)}] \equiv R_i$ is a constant on M_i), in coordinates $y_i^{n_i}$, $n_i = 1, \dots, d_i$, and

$$x \mapsto \bar{g}^{(0)}(x) = \bar{g}_{\mu\nu}^{(0)}(x) dx^\mu \otimes dx^\nu \quad (2.2)$$

yields a general, not necessarily R -homogeneous, (pseudo) Riemannian metric on \overline{M}_0 .

Below, the $\bar{g}^{(0)}$ -covariant derivative of a given function α w.r.t. x^μ is denoted by $\alpha_{;\mu}$, its partial derivative also by $\alpha_{,\mu}$, and $(\partial\alpha)(\partial\beta) := \bar{g}^{(0)\mu\nu} \alpha_{;\mu} \beta_{;\nu}$. Furthermore we use the shorthand $|g| := |\det(g_{MN})|$, $|\bar{g}^{(0)}| := |\det(\bar{g}_{\mu\nu}^{(0)})|$, and analogously for all other metrics including $g^{(i)}$, $i = 1, \dots, n$.

On \overline{M}_0 , the Laplace-Beltrami operator $\Delta[\overline{g}^{(0)}] = \frac{1}{\sqrt{|\overline{g}^{(0)}|}} \frac{\partial}{\partial x^\mu} \left(\sqrt{|\overline{g}^{(0)}|} \overline{g}^{(0)\mu\nu} \frac{\partial}{\partial x^\nu} \right)$, transforms under the conformal map $\overline{g}^{(0)} \mapsto e^{2\overline{\gamma}} \overline{g}^{(0)}$ according to

$$\begin{aligned} \Delta[e^{2\overline{\gamma}} \overline{g}^{(0)}] &= e^{-2\overline{\gamma}} \Delta[\overline{g}^{(0)}] - e^{-2\overline{\gamma}} \overline{g}^{(0)\mu\nu} \left(\Gamma[e^{2\overline{\gamma}} \overline{g}^{(0)}] - \Gamma[\overline{g}^{(0)}] \right)^\lambda_{\mu\nu} \frac{\partial}{\partial x^\lambda} \\ &= e^{-2\overline{\gamma}} \left(\Delta[\overline{g}^{(0)}] + (D_0 - 2) \overline{g}^{(0)\mu\nu} \frac{\partial \overline{\gamma}}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right), \end{aligned} \quad (2.3)$$

where Γ denotes the Levi-Civita connection.

Then, for the multidimensional metric (1.2) the Ricci tensor decomposes likewise into blocks and the corresponding Ricci curvature scalar reads

$$\begin{aligned} R[g] &= e^{-2\overline{\gamma}} R[\overline{g}^{(0)}] + \sum_{i=1}^n e^{-2\beta^i} R[g^{(i)}] - e^{-2\overline{\gamma}} \overline{g}^{(0)\mu\nu} \left((D_0 - 2)(D_0 - 1) \frac{\partial \overline{\gamma}}{\partial x^\mu} \frac{\partial \overline{\gamma}}{\partial x^\nu} \right. \\ &\quad \left. + \sum_{i,j=1}^n (d_i \delta_{ij} + d_i d_j) \frac{\partial \beta^i}{\partial x^\mu} \frac{\partial \beta^j}{\partial x^\nu} + 2(D_0 - 2) \sum_{i=1}^n d_i \frac{\partial \overline{\gamma}}{\partial x^\mu} \frac{\partial \beta^i}{\partial x^\nu} \right) \\ &\quad - 2e^{-2\overline{\gamma}} \Delta[\overline{g}^{(0)}] \left((D_0 - 1) \overline{\gamma} + \sum_{i=1}^n d_i \beta^i \right). \end{aligned} \quad (2.4)$$

Let us now set

$$f \equiv f[\overline{\gamma}, \beta] := (D_0 - 2) \overline{\gamma} + \sum_{j=1}^n d_j \beta^j, \quad (2.5)$$

where β is the vector field with the dilatonic scalar fields β^i as components. (Note that f can be resolved for $\overline{\gamma} \equiv \overline{\gamma}[f, \beta]$ if and only if $D_0 \neq 2$. The singular case $D_0 = 2$ is discussed in [9].) Then, (2.4) can also be written as

$$\begin{aligned} R[g] - e^{-2\overline{\gamma}} R[\overline{g}^{(0)}] - \sum_{i=1}^n e^{-2\beta^i} R_i &= \\ &= -e^{-2\overline{\gamma}} \left\{ \sum_{i=1}^n d_i (\partial \beta^i)^2 + (\partial f)^2 + (D_0 - 2) (\partial \overline{\gamma})^2 + 2\Delta[\overline{g}^{(0)}] (f + \overline{\gamma}) \right\} \\ &= -e^{-2\overline{\gamma}} \left\{ \sum_{i=1}^n d_i (\partial \beta^i)^2 + (D_0 - 2) (\partial \overline{\gamma})^2 - (\partial f) \partial (f + 2\overline{\gamma}) + R_B \right\}, \\ R_B &:= \frac{1}{\sqrt{|\overline{g}^{(0)}|}} e^{-f} \partial_\mu \left[2e^f \sqrt{|\overline{g}^{(0)}|} \overline{g}^{(0)\mu\nu} \partial_\nu (f + \overline{\gamma}) \right], \end{aligned} \quad (2.7)$$

where the last term will yield just a boundary contribution (2.12) to the action (2.11) below.

Let us assume all M_i , $i = 1, \dots, n$, to be connected and oriented. Then a volume form on M_i is defined by

$$\tau_i := \sqrt{|g^{(i)}(y_i)|} dy_i^1 \wedge \dots \wedge dy_i^{d_i}, \quad (2.8)$$

and the total internal space volume is

$$\mu := \prod_{i=1}^n \mu_i, \quad \mu_i := \int_{M_i} \tau_i = \int_{M_i} d^{d_i} y_i \sqrt{|g^{(i)}|}. \quad (2.9)$$

If all of the spaces M_i , $i = 1, \dots, n$ are compact, then the volumes μ_i and μ are finite, and so are also the numbers $\rho_i = \int_{M_i} d^{d_i} y \sqrt{|g^{(i)}|} R[g^{(i)}]$. However, a non-compact M_i might have infinite volume μ_i or infinite ρ_i . Nevertheless, by the R -homogeneity of $g^{(i)}$ (in particular satisfied for Einstein spaces), the ratios $\frac{\rho_i}{\mu_i} = R[g^{(i)}]$, $i = 1, \dots, n$, are just finite constants. In any case, we must tune the D -dimensional coupling constant κ (if necessary to infinity), such that, under the dimensional reduction $\text{pr} : M \rightarrow \overline{M}_0$,

$$\kappa_0 := \kappa \cdot \mu^{-\frac{1}{2}} \quad (2.10)$$

becomes the D_0 -dimensional physical coupling constant. If $D_0 = 4$, then $\kappa_0^2 = 8\pi G_N$, where G_N is the Newton constant. The limit $\kappa \rightarrow \infty$ for $\mu \rightarrow \infty$ is in particular harmless, if D -dimensional gravity is given purely by curvature geometry, without additional matter fields. If however this geometry is coupled with finite strength to additional (matter) fields, one should indeed better take care to have all internal spaces M_i , $i = 1, \dots, n$ compact. If for some homogeneous space this is a priori not the case, it often can still be achieved by factorizing this space by an appropriate finite symmetry group.

With the total dimension D , κ^2 a D -dimensional gravitational constant and Λ a D -dimensional cosmological constant we consider an action of the form

$$S = \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} \{R[g] - 2\Lambda\} + S_{\text{GHY}} + S_\Phi + S_\rho. \quad (2.11)$$

Here a (generalized) Gibbons-Hawking-York [25,26] type boundary contribution S_{GHY} to the action is taken to cancel boundary terms. Eqs.(2.6) and (2.7) show that S_{GHY} should be taken in the form

$$\begin{aligned} S_{\text{GHY}} &:= \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} \{e^{-2\overline{\gamma}} R_B\} \\ &= \frac{1}{\kappa_0^2} \int_{\overline{M}_0} d^{D_0} x \frac{\partial}{\partial x^\lambda} \left(e^f \sqrt{|\overline{g}^{(0)}|} \overline{g}^{(0)\lambda\nu} \frac{\partial}{\partial x^\nu} (f + \overline{\gamma}) \right), \end{aligned} \quad (2.12)$$

which is just a pure boundary term in form of an effective D_0 -dimensional flow through $\partial\overline{M}_0$.

Also the other additional D -dimensional action terms depend effectively only on \overline{M}_0 , like

$$S_\Phi = -\frac{1}{2} \int_M d^D z \sqrt{|g|} C(\partial\Phi, \partial\Phi) = -\frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} C_{ab} g^{MN} \partial_M \Psi^a \partial_N \Psi^b, \quad (2.13)$$

generated from a metric C on k -dimensional target space evaluated on a rescaled target vector field $\Psi := \kappa\Phi$ built from a finite number of scalar matter fields components Ψ^a , $a = 1, \dots, k$, depending only on \overline{M}_0 , and

$$S_\rho = - \int_M d^D z \sqrt{|g|} \rho \quad (2.14)$$

from a general effective matter density ρ corresponding a potential on \overline{M}_0 which may e.g. be chosen to account for the Casimir effect [27], a Freund-Rubin monopole [4], or a perfect fluid [22,23].

After dimensional reduction the action (2.11) reads

$$S = \frac{1}{2\kappa_0^2} \int_{\overline{M}_0} d^{D_0} x \sqrt{|\overline{g}^{(0)}|} e^f \left\{ R[\overline{g}^{(0)}] + (\partial f)(\partial[f + 2\overline{\gamma}]) - \sum_{i=1}^n d_i (\partial\beta^i)^2 - (D_0 - 2)(\partial\overline{\gamma})^2 - C_{ab}(\partial\Psi^a)(\partial\Psi^b) + e^{2\overline{\gamma}} \left[\sum_{i=1}^n e^{-2\beta^i} R_i - 2\Lambda - 2\kappa^2 \rho \right] \right\}, \quad (2.15)$$

where e^f is a dilatonic scalar field coupling to the D_0 -dimensional geometry on \overline{M}_0 .

According to the considerations above, due to the conformal reparametrization invariance of the geometry on \overline{M}_0 , we should fix a conformal frame on \overline{M}_0 . But then in (2.15) $\overline{\gamma}$, and with (2.5) also f , is no longer independent from the vector field β , but rather

$$\overline{\gamma} \equiv \overline{\gamma}[\beta], \quad f \equiv f[\beta]. \quad (2.16)$$

Then, modulo the conformal factor e^f , the dilatonic kinetic term of (2.15) takes the form

$$(\partial f)(\partial[f + 2\overline{\gamma}]) - \sum_{i=1}^n d_i (\partial\beta^i)^2 - (D_0 - 2)(\partial\overline{\gamma})^2 = -G_{ij}(\partial\beta^i)(\partial\beta^j), \quad (2.17)$$

with $G_{ij} \equiv {}^{(\overline{\gamma})}G_{ij}$, where

$${}^{(\overline{\tau})}G_{ij} := {}^{(\text{BD})}G_{ij} - (D_0 - 2)(D_0 - 1) \frac{\partial \overline{\tau}}{\partial \beta^i} \frac{\partial \overline{\tau}}{\partial \beta^j} - 2(D_0 - 1)d_{(i} \frac{\partial \overline{\tau}}{\partial \beta^{j)}}, \quad (2.18)$$

$${}^{(\text{BD})}G_{ij} := \delta_{ij}d_i - d_id_j. \quad (2.19)$$

For $D_0 \neq 2$, we can write equivalently $G_{ij} \equiv {}^{(f)}G_{ij}$, where

$${}^{(f)}G_{ij} := {}^{(\text{E})}G_{ij} - \frac{D_0 - 1}{D_0 - 2} \frac{\partial f}{\partial \beta^i} \frac{\partial f}{\partial \beta^j}, \quad (2.20)$$

$${}^{(\text{E})}G_{ij} := \delta_{ij}d_i + \frac{d_id_j}{D_0 - 2}. \quad (2.21)$$

For $D_0 = 1$, $G_{ij} = {}^{(\text{E})}G_{ij} = {}^{(\text{BD})}G_{ij}$ is independent of $\overline{\tau}$ and f . Note that the metrics (2.19) and (2.21) (with $D_0 \neq 2$) may be diagonalized by appropriate homogeneous linear minisuperspace coordinate transformations (see e.g. [9,17,28]) to $(\mp)(\pm)^{\delta_{1D_0}}\delta_{ij}$ respectively.

After gauging $\overline{\tau}$, setting $m := \kappa_0^{-2}$, (2.15) yields a σ -model in the form

$${}^{(\overline{\tau})}S = \int_{\overline{M}_0} d^{D_0}x \sqrt{|\overline{g}^{(0)}|} {}^{(\overline{\tau})}N^{D_0} \phi(\beta) \left\{ \frac{m}{2} {}^{(\overline{\tau})}N^{-2} \left[R[\overline{g}^{(0)}] - {}^{(\overline{\tau})}G_{ij}(\partial\beta^i)(\partial\beta^j) - C_{ab}(\partial\Phi^a)(\partial\Phi^b) \right] - {}^{(\text{BD})}V(\beta) \right\}, \quad (2.22)$$

$$\text{where} \quad {}^{(\text{BD})}V(\beta) := m \left[\Lambda + \kappa^2 \rho - \frac{1}{2} \sum_{i=1}^n R[g^{(i)}] e^{-2\beta^i} \right], \quad (2.23)$$

$${}^{(\overline{\tau})}N := e^{\overline{\tau}}. \quad (2.24)$$

Note that, the potential (2.23) and the conformal factor $\phi(\beta) := \prod_{i=1}^n e^{d_i \beta^i}$ are gauge invariant.

Analogously, the σ -model action from (2.15) gauging f can also be written as

$${}^{(f)}S = \int_{\overline{M}_0} d^{D_0}x \sqrt{|\overline{g}^{(0)}|} {}^{(f)}N^{D_0} \left\{ \frac{m}{2} {}^{(f)}N^{-2} \left[R[\overline{g}^{(0)}] - {}^{(f)}G_{ij}(\partial\beta^i)(\partial\beta^j) - C_{ab}(\partial\Phi^a)(\partial\Phi^b) \right] - {}^{(\text{E})}V(\beta) \right\}, \quad (2.25)$$

$${}^{(\text{E})}V(\beta) := m\Omega^2 \left[\Lambda + \kappa^2 \rho - \frac{1}{2} \sum_{i=1}^n R[g^{(i)}] e^{-2\beta^i} \right], \quad (2.26)$$

$${}^{(f)}N := e^{\frac{f}{D_0-2}}, \quad (2.27)$$

where the function Ω on \overline{M}_0 is defined as

$$\Omega := \phi^{\frac{1}{2-D_0}}. \quad (2.28)$$

Note that, with Ω also the potential (2.26) is gauge invariant, and the dilatonic target-space, though not even conformally flat in general, is flat for constant f .

In fact, Eqs. (2.22)-(2.24) and (2.25)-(2.27) show that there are at least two special frames.

The first one corresponds to the gauge $\bar{\gamma} \stackrel{!}{=} 0$. In this case $(\bar{\gamma})N = 1$, the minisuperspace metric (2.18) reduces to the Minkowskian (2.19), the dilatonic scalar field becomes proportional to the internal space volume, $e^{f[\beta]} = \phi(\beta) = \prod_{i=1}^n e^{d_i \beta^i}$, and (2.22) describes a generalized σ -model with conformally Minkowskian target space. The Minkowskian signature implies a negative sign in the dilatonic kinetic term. This frame is usually called the Brans-Dicke one, because $\phi = e^f$ here plays the role of the Brans-Dicke scalar field. Our effective theory following from multidimensional cosmology [9] takes a generalized Brans-Dicke form.

The second distinguished frame corresponds to the gauge $f \stackrel{!}{=} 0$, where $\bar{\gamma} = \frac{1}{2-D_0} \sum_{i=1}^n d_i \beta^i$ is well-defined only for $D_0 \neq 2$. In this case $(f)N = 1$, the minisuperspace metric (2.20) reduces to the Euclidean (2.21), and (2.25) describes a self-gravitating σ -model with Euclidean target space. Hence all dilatonic kinetic terms have positive signs. This frame is usually called the Einstein one, because it describes an effective D_0 -dimensional Einstein theory with additional minimally coupled scalar fields. For multidimensional geometries with $D_0 = 2$ the Einstein frame fails to exist, which reflects the well-known fact that two-dimensional Einstein equations are trivially satisfied without implying any dynamics.

For $D_0 = 1$, the action of both (2.22) and (2.25) was shown in [28] (and previously in [29,30]) to take the form of a classical particle motion on minisuperspace, whence different frames correspond are just related by a time reparametrization. More generally, for $D_0 \neq 2$ and $(\bar{M}_0, \bar{g}(0))$ a vacuum space-time, the σ -model (2.25) with the gauge $f \stackrel{!}{=} 0$ describes the dynamics of a massive $(D_0 - 1)$ -brane within a potential (2.26) on its target minisuperspace.

Before concluding this chapter, let us point out that besides the Brans-Dicke and the Einstein gauge, which are the main topic of this paper, there might be further gauges of interest for particular physical features.

From (2.15) we see that, there exists another similarly distinguished frame, namely the one corresponding to the gauge $f \stackrel{!}{=} -2\overline{\gamma} = \frac{2}{D_0} \sum_{i=1}^n d_i \beta^i$, in which, as for the Einstein frame, the kinetic term $(\partial f)(\partial[f + 2\overline{\gamma}])$ carrying the anomalous sign vanishes, whence the target minisuperspace carries a true (not just a pseudo) metric corresponding to a non-negative kinetic contribution to the action. In this gauge the potential terms decouple from the dilatonic field f , although the latter still couples to the kinetic terms.

Of course the choice of any a priori prescribed action strongly affects the “natural” choice of frame. For different theories we can introduce different ”natural” gauges.

For example starting from an D -dimensional effective string action which includes besides the dilaton also a massless axion there is a so-called “axion” gauge [31] which decouples the axion from the dilaton field.

We conclude by emphasizing again that for theories (1.1) with action of the type (2.11) there exist compelling physical arguments in favor of the Einstein frame. Therefore we will now investigate how to generate solutions in this frame.

III. GENERATING SOLUTIONS IN THE EINSTEIN FRAME

In the following we denote the external space-time metric $\overline{g}^{(0)}$ in the Brans-Dicke frame with $\overline{\gamma} \stackrel{!}{=} 0$ as $\overline{g}^{(\text{BD})}$ and in the Einstein frame with $f \stackrel{!}{=} 0$ as $\overline{g}^{(\text{E})}$. It can be easily seen that they are connected with each other by a conformal transformation

$$\overline{g}^{(\text{E})} \mapsto \overline{g}^{(\text{BD})} = \Omega^2 \overline{g}^{(\text{E})} \quad (3.1)$$

with Ω from (2.28).

Let us now consider the space time foliation $\overline{M}_0 = \mathbb{R} \times M_0$ where $g^{(0)}$ is a smooth homogeneous metric on M_0 . Under any projection $\text{pr}_0 : \overline{M}_0 \rightarrow \mathbb{R}$ a consistent pullback of the metric $-e^{2\gamma(\tau)} d\tau \otimes d\tau$ from $\tau \in \mathbb{R}$ to $x \in \text{pr}_0^{-1}\{\tau\} \subset \overline{M}_0$ is given by

$$\overline{g}^{(\text{BD})}(x) := -e^{2\gamma(\tau)} d\tau \otimes d\tau + e^{2\beta^0(x)} g^{(0)}. \quad (3.2)$$

For spatially (metrically-)homogeneous cosmological models as considered below all scale factors $a_i := e^{\beta^i}$, $i = 0, \dots, n$ depend only on $\tau \in \mathbb{R}$.

With (3.2) and (3.1), Eq. (1.2) reads

$$\begin{aligned} g &= -e^{2\gamma(\tau)} d\tau \otimes d\tau + a_0^2 g^{(0)} + \sum_{i=1}^n e^{2\beta^i} g^{(i)} \\ &= -dt_{\text{BD}} \otimes dt_{\text{BD}} + a_{\text{BD}}^2 g^{(0)} + \sum_{i=1}^n e^{2\beta^i} g^{(i)} \\ &= -\Omega^2 dt_{\text{E}} \otimes dt_{\text{E}} + \Omega^2 a_{\text{E}}^2 g^{(0)} + \sum_{i=1}^n e^{2\beta^i} g^{(i)}, \end{aligned} \quad (3.3)$$

where $a_0 := a_{\text{BD}}$ and a_{E} are the external space scale factor functions depending respectively on the cosmic synchronous time t_{BD} and t_{E} in the Brans-Dicke and Einstein frame. With (2.28) the latter is related to the former by

$$a_{\text{E}} = \Omega^{-1} a_{\text{BD}} = \left(\prod_{i=1}^n e^{d_i \beta^i} \right)^{\frac{1}{D_0-2}} a_{\text{BD}}, \quad (3.4)$$

and the cosmic time of the Einstein frame is given by

$$|dt_{\text{E}}| = |\Omega^{-1} e^{\gamma} d\tau| = \left| \left(\prod_{i=1}^n e^{d_i \beta^i} \right)^{\frac{1}{D_0-2}} dt_{\text{BD}} \right|. \quad (3.5)$$

As a consequence of the arguments mentioned in the introduction, t_{E} will be considered below as the physical time. The presently best known (spatially homogeneous) cosmological solutions with a metric structure given by (1.2) and (3.2) were found in the Brans-Dicke frame (see e.g. [21–24] and an extensive list of references there). Most of them are described most simply within one of the following two systems of target space coordinates. We set

$$q := \sqrt{\frac{D-1}{D-2}}, \quad p := \sqrt{\frac{d_0-1}{d_0}}. \quad (3.6)$$

With $\Sigma_k = \sum_{i=k}^n d_i$, the first coordinate system [32] is related to β^i , $i = 0, \dots, n$, as

$$\begin{aligned} z^0 &:= q^{-1} \sum_{j=0}^n d_j \beta^j, \\ z^i &:= [d_{i-1} / \Sigma_{i-1} \Sigma_i]^{1/2} \sum_{j=i}^n d_j (\beta^j - \beta^{i-1}), \quad i = 1, \dots, n, \end{aligned} \quad (3.7)$$

and the second one [33] as

$$\begin{aligned}
v^0 &:= p^{-1} \left(\sum_{j=0}^n d_j \beta^j - \beta^0 \right), \\
v^1 &:= p^{-1} [(D-2)/d_0 \Sigma_1]^{1/2} \sum_{j=1}^n d_j \beta^j, \\
v^i &:= [d_{i-1}/\Sigma_{i-1} \Sigma_i]^{1/2} \sum_{j=i}^n d_j (\beta^j - \beta^{i-1}), \quad i = 2, \dots, n,
\end{aligned} \tag{3.8}$$

In both of this minisuperspace coordinates the target space Minkowski metric G is given in form of the standard diagonal matrix $G_{ij} := (-)^{\delta_{0i}} \delta_{ij}$. The two coordinates are related by a Lorentz boost in the (01)-plane.

In coordinates (3.7) some known solutions (see e.g. [24,34,35]) take the form

$$a_i = A_i (e^{qz^0})^{\frac{1}{D-1}} e^{\alpha^i \tau}, \quad i = 0, \dots, n, \tag{3.9}$$

where parameters α^i satisfy conditions

$$\begin{aligned}
\sum_{i=0}^n d_i \alpha^i &= 0, \\
\sum_{i=0}^n d_i (\alpha^i)^2 &= 2\epsilon
\end{aligned} \tag{3.10}$$

and ϵ is a non-negative parameter.

In coordinates (3.8) some known solutions (see e.g. [23,36]) take the form

$$\begin{aligned}
a_0 &= A_0 (e^{pv^0})^{\frac{1}{d_0-1}} e^{\alpha^0 \tau}, \\
a_i &= A_i e^{\alpha^i \tau}, \quad i = 1, \dots, n,
\end{aligned} \tag{3.11}$$

where parameters α^i satisfy conditions

$$\begin{aligned}
\sum_{i=0}^n d_i \alpha^i &= \alpha^0, \\
\sum_{i=0}^n d_i (\alpha^i)^2 &= (\alpha^0)^2 + 2\epsilon
\end{aligned} \tag{3.12}$$

and ϵ is a non-negative parameter.

Explicit expressions for functions $z^0 \equiv z^0(\tau)$ and $v^0 \equiv v^0(\tau)$ depend on the details of the particular models.

Solutions of the form (3.9) and (3.11) were found in the harmonic time gauge $\gamma \stackrel{!}{=} \sum_{j=0}^n d_j \beta^j$, where τ is the harmonic time of the Brans-Dicke frame. Equation (3.7) shows that the coordinate z^0 is related to the dynamical part of the total spatial volume in the BD frame: $v := e^{qz^0} = \prod_{i=0}^n a_i^{d_i}$.

Relations (3.4) and (3.8) between the different minisuperspace coordinates imply that

$$(d_0 - 1)\beta_E^0 = (d_0 - 1)\beta^0 + \sum_{j=1}^n d_j \beta^j = pv^0, \quad (3.13)$$

which shows that the coordinate v^0 is proportional to the logarithmic scale factor of external space in the Einstein frame: $a_E := e^{\beta_E^0}$.

Thus target space coordinates z have the most natural interpretation in the Brans-Dicke frame, whereas target space coordinates v are better adapted to the Einstein frame.

Via (3.13) synchronous time in the Einstein frame is related to harmonic time τ in the Brans-Dicke frame by integration of (3.5) with integration constant c to

$$|t_E| + c = \int (e^{pv^0})^{d_0/d_0-1} d\tau = \int a_E^{d_0} d\tau. \quad (3.14)$$

Thus the physical metric of external space-time reads

$$g^{(E)} = -a_E^{2d_0} d\tau \otimes d\tau + a_E^2 g^{(0)}, \quad (3.15)$$

where for solutions (3.9)

$$a_E = \left[\frac{(e^{qz^0})^{\frac{1}{q^2}}}{A_0 e^{\alpha^0 \tau}} \right]^{\frac{1}{d_0-1}}, \quad (3.16)$$

and for solutions (3.11)

$$a_E = (e^{pv^0})^{\frac{1}{d_0-1}}. \quad (3.17)$$

Expressions for the internal scale factors are not affected. In Eqs. (3.15) to (3.17) the time τ is the harmonic one from the Brans-Dicke frame. The transformation to synchronous time in the Einstein frame is provided by Eq. (3.14). Once z^0 or v^0 is known as a function of τ ,

explicit expressions can be given. However these functions depends on the concrete form of the cosmological model (see [21]- [24], [33]- [36]).

Above we obtained a general prescription for the generation of solutions in the Einstein frame from already known ones in the Brans-Dicke frame. It can easily be seen that the behavior of the solutions in both of these frames is quite different. Let us demonstrate this explicitly by the example of a generalized Kasner solution.

IV. SOLUTIONS IN ORIGINAL FORM

Let $t := t_{\text{BD}}$ be the synchronous time of the Brans-Dicke frame, and \dot{x} denote the derivative of x with respect to t .

The well-known Kasner solution [37] describes a 4-dimensional anisotropic space-time with the metric

$$g = -dt \otimes dt + \sum_{i=1}^3 t^{2p_i} dx^i \otimes dx^i \quad (4.1)$$

where the p_i are constants satisfying

$$\sum_{i=1}^3 p_i = \sum_{i=1}^3 (p_i)^2 = 1. \quad (4.2)$$

It is clear that a multidimensional generalization of this solution is possible for a manifold (1.1) with Ricci flat factor spaces (M_i, g_i) , $i = 0, \dots, n$. Particular solutions generalizing (4.1) with (4.2) were obtained in many papers [38]- [43]. More general solutions for an arbitrary number of d_i -dimensional tori were found in [44] and generalized to the case of a free minimally coupled scalar field Φ in [45]. In the latter case there are two classes of solutions.

A first class represents namely Kasner-like solutions. None of these is contained in the hypersurface

$$\sum_{i=0}^n d_i \dot{\beta}^i = 0 \quad (4.3)$$

of constant spatial volume. With c and $a_{(0)i}$, $i = 0, \dots, n$ integration constants, in the Brans-Dicke synchronous time gauge such a solution reads

$$a_i = a_{(0)i} t^{\bar{\alpha}^i}, \quad i = 0, \dots, n, \quad (4.4)$$

$$\Phi = \ln t^{\bar{\alpha}^{n+1}} + c, \quad (4.5)$$

where the $\bar{\alpha}^i$ fulfill the conditions

$$\begin{aligned} \sum_{i=0}^n d_i \bar{\alpha}^i &= 1, \\ \sum_{i=0}^n d_i (\bar{\alpha}^i)^2 &= 1 - (\bar{\alpha}^{n+1})^2. \end{aligned} \quad (4.6)$$

Without an additional non-trivial scalar field Φ , i.e. for $\bar{\alpha}^{n+1} = 0$, these conditions become analogous to (4.2)

$$\sum_{i=0}^n d_i \bar{\alpha}^i = \sum_{i=0}^n d_i (\bar{\alpha}^i)^2 = 1. \quad (4.7)$$

Solution (4.4) describes a universe with increasing total spatial volume

$$v \sim \prod_{i=0}^n a_i^{d_i} \sim t \quad (4.8)$$

and decreasing Hubble parameter for each factor space

$$h_i := \frac{1}{a_i} \frac{da_i}{dt} = \frac{\bar{\alpha}^i}{t}, \quad i = 0, \dots, n. \quad (4.9)$$

In the case of imaginary scalar field ($(\bar{\alpha}^{n+1})^2 < 0$) factor spaces with $\bar{\alpha}^i > 1$ undergo a power law inflation. The absence of a non-trivial scalar field, i.e. $\Phi \equiv 0$, implies (except for $d_0 = \bar{\alpha}^0 = 1$, $\bar{\alpha}^i = 0$, $i = 1, \dots, n$) that $|\bar{\alpha}^i| < 1$ for $i = 0, \dots, n$. In [46] it was shown that after a transformation $t \rightarrow t_0 - t$ (reversing the arrow of time) factor spaces with $\bar{\alpha}^i < 0$ can be interpreted as inflationary universes with scale factors $a_i \sim (t_0 - t)^{\bar{\alpha}^i}$ with $\bar{\alpha}^i < 0$ growing at an accelerated rate $\ddot{a}_i/a_i > 0$.

A second (more special) class of solutions is confined to the hyperplane (4.3) in momentum space. In this case (in the Brans-Dicke frame) harmonic and synchronous time coordinates coincide and solutions read

$$a_i = a_{(0)i} e^{b^i t}, \quad i = 0, \dots, n, \quad (4.10)$$

$$\Phi = b^{n+1} t + c, \quad (4.11)$$

where c is a integration constant and the constants b^i satisfy

$$\begin{aligned} \sum_{i=0}^n d_i b^i &= 0, \\ \sum_{i=0}^n d_i (b^i)^2 + (b^{n+1})^2 &= 0. \end{aligned} \quad (4.12)$$

The latter relation shows that these solutions are only possible if Φ is an imaginary scalar field with $(b^{n+1})^2 < 0$.

The inflationary solution (4.10) describes a universe with constant total spatial volume

$$v \sim \prod_{i=0}^n a_i^{d_i} = \prod_{i=0}^n a_{(0)i}^{d_i}, \quad (4.13)$$

and a nonzero but constant Hubble parameter

$$h_i = \frac{1}{a_i} \frac{da_i}{dt} = b^i, \quad i = 0, \dots, n, \quad (4.14)$$

for each factor space. This is a particular case of a steady state universe where stationarity of matter energy density in the whole universe is maintained due to redistribution of matter between contracting and expanding parts (factor spaces) of the universe (matter density in the whole universe is constant due to the constant volume). This is unlike the original steady-state theory [47], where a continuous creation of matter is required in order to stabilize matter density, which then necessitates a deviation from Einstein theory. In [28] the inflationary solution was generalized for the case of a σ -model with k -dimensional target vectors Φ rather than a single scalar field.

V. SOLUTIONS IN THE EINSTEIN FRAME

Let us now transform the solutions (4.4), (4.5) and (4.10),(4.11) above to the Einstein frame, using the general prescription from Sec. III.

We first consider the Kasner-like solution (4.4), where (2.28) determines the conformal factor as

$$\Omega^{-1} = \left(\prod_{i=1}^n e^{d_i \beta^i} \right)^{\frac{1}{D_0-2}} = C_1 t^{(1-d_0 \bar{\alpha}^0)/(d_0-1)} \quad (5.1)$$

with

$$C_1 := \left(\prod_{i=1}^n a_{(0)i}^{d_i} \right)^{\frac{1}{D_0-2}}. \quad (5.2)$$

As it was noted above, the conformal transformation to the Einstein frame does not exist for $D_0 = 2$ ($d_0 = 1$). In the special case of $\bar{\alpha}^0 = \frac{1}{d_0}$ the conformal factor Ω is constant, and both frames represent the same connection, hence the same geometry.⁶ Even in this case, (5.2) is still divergent for $d_0 = 1$.

The external space scale factor in the Einstein frame (physical scale factor of the external space) is defined by formula (3.4) which for (5.1) reads

$$a_E = \Omega^{-1} a_{BD} = \bar{a}_0 t^{(1-\bar{\alpha}^0)/(d_0-1)}, \quad (5.3)$$

where $\bar{a}_0 := C_1 a_{(0)0}$. At $\bar{\alpha}^0 = \frac{1}{d_0}$ the (external space) scale factor $a_E = \bar{a}_0 t^{\bar{\alpha}^0} \sim a_{BD}$ has the same behavior in both frames which is just what one expects for constant Ω .

So the physical metric of the external space-time reads

$$\bar{g}^{(E)} = -\Omega^{-2} dt \otimes dt + a_E^2 g^{(0)} = -dt_E \otimes dt_E + a_E^2 g^{(0)}, \quad (5.4)$$

where Ω^{-1} and a_E are given by equations (5.1) and (5.3) respectively, and t is given synchronous time in the Brans-Dicke frame connected with synchronous time in the Einstein frame via (3.5). Putting the integration constant to zero we obtain

$$t = C_2 t_E^{(d_0-1)/d_0(1-\bar{\alpha}^0)}, \quad (5.5)$$

⁶Here is meant the geometry as given by the connection. Locally at $x \in \overline{M}_0$ this is just the $\text{End}(T_x \overline{M}_0)$ -valued Riemannian curvature 2-form.

where $C_2 = \left[C_1^{-1} \frac{1-\bar{\alpha}^0}{d_0-1} d_0 \right]^{(d_0-1)/d_0(1-\bar{\alpha}^0)}$. The value $\bar{\alpha}^0 = 1$ is a singular one. It can be seen from (4.6) that $|\bar{\alpha}^i| < 1$, $i = 0, \dots, n+1$ when the scalar field is real. The value $\bar{\alpha}^0 = 1$ may appear only in the case of an imaginary scalar field. (5.3) shows that, in this case a_E is a constant. In the case $\bar{\alpha}^0 \neq 1$ the generalized Kasner-like solutions in the Einstein frame take the form

$$a_{i,E} = \tilde{a}_i t_E^{\tilde{\alpha}^i}, \quad i = 0, \dots, n, \quad (5.6)$$

$$\Phi = \tilde{\alpha}^{n+1} \ln t_E + c. \quad (5.7)$$

Here and in the following $a_{0,E} := a_E(t_E)$, $a_{i,E} := a_i(t_E)$, $i = 1, \dots, n$, are given as functions of t_E , while \tilde{a}_i , $i = 0, \dots, n$, and c are constants. In (5.6) and (5.7) the powers $\tilde{\alpha}^i$ are defined as

$$\begin{aligned} \tilde{\alpha}^0 &:= \frac{1}{d_0} \\ \tilde{\alpha}^i &:= \frac{d_0 - 1}{d_0(1 - \bar{\alpha}^0)} \bar{\alpha}^i, \quad i = 1, \dots, n+1, \end{aligned} \quad (5.8)$$

with $\bar{\alpha}^i$, $i = 0, \dots, n+1$, satisfying relations (4.6). Hence in contrast to (4.4) there is no freedom in the choice of the power $\tilde{\alpha}^0$. For example at $d_0 = 3$ one obtains a physical external space scale factor $a_E = t_E^{1/3}$, i.e. the external space (M_0, g_0) behaves like a Friedmann universe filled with ultra stiff matter (which is equivalent to a minimally coupled scalar field).

Let us emphasize here once more that in the present approach the physical theory is modeled as a D_0 -dimensional effective action with the space-time metric (5.4) in the Einstein frame ($f = 0$). All internal spaces are displayed in the external space-time as scalar fields, leading to a D_0 -dimensional self-gravitating σ -model with self-interaction [9].

Let us transform now the inflationary solution (4.10) to the Einstein frame. For this solution the conformal factor and the external space scale factor read

$$\Omega^{-1} = C_1 \exp \left(-\frac{d_0 b^0}{d_0 - 1} t \right), \quad (5.9)$$

$$a_E = \bar{a}_0 \exp \left(-\frac{b^0}{d_0 - 1} t \right), \quad (5.10)$$

where C_1 is defined by (5.2) and $\bar{a}_0 = C_1 a_{(0)0}$. Note that the conformal transformation (5.9) breaks down for $D_0 = 2$ ($d_0 = 1$). This happens even in the special case of $b^0 = 0$. For the latter, Ω is constant, whence the connection and its geometry represented by both frames are the same. Here, the external space is static in both of them.

For $b^0 \neq 0$, synchronous times in the Brans-Dicke and Einstein frames are related by

$$t = \frac{d_0 - 1}{d_0 b^0} \ln(C_2 t_E^{-1}), \quad (5.11)$$

where (taking a relative minus sign in (3.5)) $C_2 = C_1 \frac{d_0 - 1}{d_0 b^0}$. Thus in the Einstein frame scale factors have power-law behavior

$$a_{i,E} = \tilde{a}_i t_E^{\tilde{\alpha}^i}, \quad i = 0, \dots, n, \quad (5.12)$$

with

$$\begin{aligned} \tilde{\alpha}^{(0)} &:= \frac{1}{d_0} \\ \tilde{\alpha}^{(i)} &:= -\frac{d_0 - 1}{d_0} \frac{b^i}{b^0} \quad i = 1, \dots, n. \end{aligned} \quad (5.13)$$

Similar as for the Kasner-like solution, the inflationary solution transformed to the Einstein frame has no freedom in choice of the power $\tilde{\alpha}^{(0)}$. The external space scale factor behaves as $a_{0,E} \sim t_E^{1/d_0}$ (compare also (5.6) and (5.8)). The scalar field reads

$$\Phi = \tilde{\alpha}^{n+1} \ln t_E + c, \quad \tilde{\alpha}^{n+1} := -\frac{d_0 - 1}{d_0} \frac{b^{n+1}}{b^0}. \quad (5.14)$$

Using (4.12) we obtain the sum rules

$$\begin{aligned} \sum_{i=0}^n d_i \tilde{\alpha}^i &= d_0, \\ (\tilde{\alpha}^{n+1})^2 &= 2 - d_0 - \sum_{i=0}^n d_i (\tilde{\alpha}^i)^2 < 0, \end{aligned} \quad (5.15)$$

whence the scalar field is imaginary.

The main lesson we learned in this section is the following: The dynamical behavior of scale factors and scalar fields strongly depends on the choice of the frame. For example in

the case of solutions originating from the Kasner and inflationary solutions of Sec. IV the external space scale factor in the Einstein frame behaves as t_E^{1/d_0} (except for the cases $\bar{\alpha}^0 = 1$ and $b^0 = 0$ where a_E is a constant). In this case there is no inflation of the external space, neither exponential nor power law (with power larger than 1). In contrast to the conclusions drawn in [46] for the Kasner solutions in the Brans-Dicke frame, inversion of arrow of time t_E in the Einstein frame does not lead to inflation of the external space because of power $1/d_0 > 0$.

VI. CONCLUSIONS

We started from a higher-dimensional cosmological model based on a smooth manifold of topology (1.1) with a multidimensional geometry given by a metric ansatz (1.2).

Then, an Einstein theory in higher dimension D can be reduced to an effective model in lower dimension D_0 . This is a (generalized) σ -model with conformally flat target space. With a purely geometrical dilaton field f , it provides a natural generalization for the well-known Brans-Dicke theory.

In the Introduction we gave several reasons which suggest that Einstein frame with $f = 0$ should be the preferred frame for a more direct physical interpretation of the model under consideration. This necessitates that, before a physical interpretation can be given, solutions previously obtained in the Brans-Dicke frame should first be transformed to the Einstein frame.

Typical solutions for considered models in Brans-Dicke frame have a general structure described either by (3.9) or (3.11). For solutions of this type the transformation to Einstein frame is given by (3.16) and (3.17) respectively. The qualitative difference induced by the distinct functions z^0 and v^0 respectively necessitates a separate treatment of these two classes. In any case, solutions to a given model in the Einstein frame show a different qualitative behavior from the corresponding solution in the Brans-Dicke frame.

We demonstrated this explicitly on the example of the generalized Kasner solution (4.4)

(and exceptional inflationary solutions (4.10)). With respect to the proper time in Einstein frame, the external space scale factor $a_{0,E}$ has a surprisingly simple and definite root law behavior $a_{0,E} \sim t_E^{1/d_0}$ (except for the case of an exotic imaginary scalar field where $a_{0,E}$ may be constant). Hence this model does not admit inflation of the external space in Einstein frame. This contrasts investigations [46] performed in the Brans-Dicke frame.

Similarly the transformation of all other known solutions can give rise to new surprising results.

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